BUCKLING OF ROTATIONALLY RESTRAINED LAMINATED COMPOSITE FLAT PANELS UNDER MULTIAXIAL LOADINGS

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ABSTRACT

In airframe structures, buckling is an important design criterion and must be well determined through an accurate methodology. Generally, simply supported boundary condition is considered in structure sizing. This approach leads to very conservative buckling allowable and consequently weight and cost penalty. To overcome this issue the current paper focuses on the buckling behavior of stiffened symmetrically laminated composite plates under shear and in-plane loadings, using an energy approach. To incorporate the stiffness induced by bordering structural elements, the stiffeners, rotational elastic restraints have been foreseen acting at plate edges. By varying the restraint stiffnesses, a wide range of boundary conditions from simply supported to clamped are definable. To make the formulation valid for moderately thick laminates a first order shear deformation displacement field has been used for the derivation of the total potential energy expression. The Ritz solution was then implemented to determine the equilibrium condition and to obtain the eigenvalue problem. The classical simply supported trigonometric displacement functions were used and the restraints were applied through adding the potential energy of rotational springs to the total potential energy expression as a penalty parameter. The results of the developed solution were then evaluated through comparison with the results obtained by finite element analysis, and an acceptable agreement was observed. Using the developed methodology a parametric study was carried out and the effect of the influential parameters such as plate aspect ratio, stack-up sequence, loading and restraint stiffness on the critical buckling loads of a family of laminates were studied. In the parametric studies equal restraint stiffness was introduced at the opposite edges which was proportional to the bending stiffness of the plate in the relevant direction. To present the results in the most meaningful and repeatable manner, non-dimensional parameters were used in parametric studies.

1 INTRODUCTION

One major design criterion of laminated composite panels is their buckling strength. In classical buckling analysis of plates or shells, the boundary conditions taken into consideration are usually either simply supported or clamped boundary conditions. However, in real world structures, panels are usually connected to other structural elements that impose additional stiffness to them and affect their stability characteristics. The effect of bordering structural elements are usually incorporated in structural analyses by adding elastic restraints at the boundaries of the panels under investigation. Diverse methodologies have been developed for studying the stability characteristics of laminated restrained panels and numerous parametric studies have been carried out. As one of the earliest works, Simitses and Giri [1] developed a methodology for buckling analysis of rotationally restrained orthotropic panels under uniaxial uniform compression using Galerkin method. They also performed parametric studies and found the ratio of Young's moduli to restraint stiffness as the most influential parameter in buckling of rotationally restrained panels.

Diverse research works have been carried out on restrained composite panels by the use of Galerkin method. Jaberzadeh et al [2] studied the buckling of elastically restrained rectangular plates to develop a methodology for modeling stiffened panels or beam flanges. With the same goal, Vescovini et al [3]obtained a closed form formula for buckling of orthotropic restrained panels. Chen et al [4] used a weighted combination of simply supported and clamped shape functions as the displacement approximation and performed a study of the influential parameters.

The Ritz method and the principle of minimum total potential energy have also been widely presented in the literature as powerful approaches for buckling analysis of restrained panels. Qiao et al [5] studied the local buckling of rectangular composite plates loaded non-uniformly on non-restrained edges. They once again tackled the same problem [6] using a linear combination of the simply supported and clamped displacement fields. Qiao et al [7] also solved the same problem with different combination of boundary conditions. Mittelstedt [8] also used the Ritz method for buckling analysis of symmetrically laminated rectangular plate under linearly varying compressive load. Shan and Qiao [9] used the Ritz method with modified shape functions to investigate the buckling problem of laminated plates with diverse combinations of boundary conditions from simply supported to clamped. Qia et al [10] used the Ritz method to develop closed form solutions for different combinations of free, simply supported and restrained boundary conditions and found the closed form solution in excellent agreement with numerical solutions. Villarreal et al [11] analytically studied the buckling problem of orthotropic laminated plates with different boundary conditions under biaxial loading. It was concluded that taking into account the restrained boundary conditions, a reduction in design conservatism can be achieved. Stamatelos et al [12] also adapted the Ritz method for investigating buckling and post buckling behavior of isotropic and symmetrically laminate plates with arbitrary boundary conditions.

Considering the inflexibility of the Ritz method with regard to boundary conditions, penalty function method has been used in some works to impose various boundary conditions [13], [14]. Applying boundary conditions through penalty functions makes it possible to take benefit of functions with faster convergence rate.

Most of the research works conducted in the area of restrained panels have used either the Ritz or Galerkin methods, however in some works other analytical approaches have been followed. Chai et al [15] used a semi-energy method to study the stability of symmetrically laminated composites with restrained edges. They investigated the effect of buckle half wave lengths and unloaded edge restraints. Bank et al [16] used the stability differential equation and studied the buckling problem of orthotropic laminates under uniaxial compression with a specific combination of edge conditions. Qiao et al[17] analytically studied the buckling problem of discrete laminated panels and panels of fiber-reinforced plastic (FRP) structural shapes and developed a formulation for explicitly calculating restraint coefficients for I and box sections.

Apart from analytical approaches, some semi-analytical and numerical investigations of restrained laminated plates have also been carried out. Examples of such works are, the use spline finite strip method by Mizusawa et al [18] and by Chen and Qiao [19]. Housner and Stein[20] also developed a finite difference program for calculation of buckling load of variable thickness orthotropic panels.

Majority of published works in the restrained plate buckling literature are based on the classical plate theory. In the current work a first order shear deformation theory is used. The implemented theory makes the solution valid for moderately thick laminates. Using the developed formulation a parametric study on a specific class of laminates has been carried out and the results are demonstrated in terms of non-dimensional parameters.

2 Formulation

In order to model the buckling behavior of curved laminated composite plates, the Ritz method which is based on the principle of minimum total potential energy is used. The formulation is developed based on a first-order shear deformation displacement field.

2.1 Kinematics of flat plate

According to the first-order shear deformation theory the displacements are assumed to be of the form

$$u = u_0(x,y) + z\phi_x \tag{1}$$
$$v = v_0(x,y) + z\phi_y \tag{2}$$
$$w = w_0\,(x,y) \tag{3}$$

where u and v are in-plane displacements and w is the out-of-plane displacement. The parameters denoted by 0 are the displacements associated to the mid-plane of the laminate. ϕ_x and ϕ_y are the rotations of the normal cross-section around Y and X axes respectively.
According to the displacement field introduced above, the linear mid-plane strains and plate curvatures within the laminate are expressed in terms of displacements as

$$\{\epsilon\} = \left[\varepsilon_x^0\ \varepsilon_y^0\ \gamma_{xy}^0\ k_x\ k_y\ k_{xy}\ \right]^T = \left[\frac{\partial u_0}{\partial x}\ \ \frac{\partial v_0}{\partial y}\ \ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\ \ \frac{\partial \varphi_x}{\partial x}\ \ \frac{\partial \varphi_y}{\partial y}\ \ \frac{\partial \varphi_y}{\partial x} + \frac{\partial \varphi_x}{\partial y}\right]^T \tag{4}$$

The out-of-plane shear strains are

$$\{\Gamma\} = \left[\gamma_{yz}\ \ \gamma_{xz}\right]^T = \left[\varphi_y + \frac{\partial w_0}{\partial y}\ \ \ \varphi_x + \frac{\partial w_0}{\partial x}\ \right]^T \tag{5}$$

Considering the nonlinear nature of the buckling phenomenon nonlinear in-plane strains are also used in the buckling analysis. These nonlinear strains denoted by superscript ′ are expressed as

$$\epsilon^{'} = \left[\varepsilon_x^{'}\ \ \varepsilon_y^{'}\ \ \gamma_{xy}^{'}\right]^T = \left[\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\ \ \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\ \ \left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial y}\right)\right]^T \tag{6}$$

2.2 Constitutive equation

Considering that the thickness of the plate is h and it consists of several sublayers of equal thickness, in the case where the origin of the coordinate system is located on the mid-plane of the laminate and Z is the plate normal, the elements of the stiffness matrix of the laminate are obtained using the formula

$$\left(\mathrm{A_{ij}}, \mathrm{B_{ij}}, \mathrm{D_{ij}}\right) = \int_{-h/2}^{h/2} Q_{i,j}(1, Z, Z^2)dz \tag{7}$$

where Q_{ij} are the elements of the reduced plane stress stiffness matrix of each sublayer.
In a similar manner the elements of the stiffness matrix related to transverse shear deformation are calculated through

$$\mathrm{H_{ij}} = k\int_{-h/2}^{h/2} C_{i,j}\,dz \quad for \quad i = 4{,}5 \tag{8}$$

where C is the complete stiffness matrix of each ply and k is the shear correction factor which has been assumed to be 2/3 throughout the present work.
The constitutive equation used here is that of shear deformation laminate theory as follows:

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = [S]\{\epsilon\} \tag{9}$$

Where
$[S] = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix}$ and $\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} N_x \ N_y \ N_{xy} \ M_x \ M_y \ M_{xy} \end{bmatrix}^T$
And N_x, N_y and N_{xy} are membrane force resultants, and M_x, M_y and M_{xy} are the moments around Y and X and the twisting moment respectively.
Subsequently the constitutive equation for transverse shear is as follows:

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = [H]\{\Gamma\} \tag{10}$$

where Q_x and Q_y are resultant shear forces on XZ and YZ cross-sections respectively.

2.3 *Total potential energy*

Considering a and b as the plate dimensions parallel to X and Y axes respectively, the total potential energy of the plate is expressed as

$$\Pi = U + U_r + V \tag{11}$$

where U is the strain potential energy, U_r is the potential energy of rotational springs and V is the potential energy associated to stress stiffening.
The strain energy of a plate with aforementioned geometrical dimensions is given by

$$U = \frac{1}{2}\left\{ \iint\limits_{0\,0}^{a\,b} \{\epsilon\}^T [S]\,\{\epsilon\} dxdy + \iint\limits_{0\,0}^{a\,b} \{\Gamma\}^T [H]\,\{\Gamma\} dxdy \right\} \tag{12}$$

In equation (12) the first integral is the potential energy of in-plane strains and the second integral represents the potential energy of the out-of-plane shearing deformation. The expression for the potential energy of rotational springs is

$$U_r = \frac{1}{2}\left\{ \int_0^b [K_{x0}(\varphi_x^2|_{x=0}) + K_{xa}(\varphi_x^2|_{x=a})]\,dy + \int_0^a \left[K_{y0}(\varphi_y^2|_{y=0}) + K_{yb}(\varphi_y^2|_{y=b})\right] dx \right\} \tag{13}$$

Where K_{x0}, K_{xa}, K_{y0} and K_{yb} are rotational stiffness of springs as shown in figure (1).
The energy associated to stress stiffening which couples the in-plane stress field to the out-of-plane displacement is

$$V = \iint_{0\,0}^{a\,b} \{N^0\}^T \{\epsilon'\} \tag{14}$$

Where $\{N^0\} = \left[N_x^0\ N_y^0\ N_{xy}^0\right]^T$ is the pre-buckled stress which here is assumed to be equal to the applied load at the boundaries.

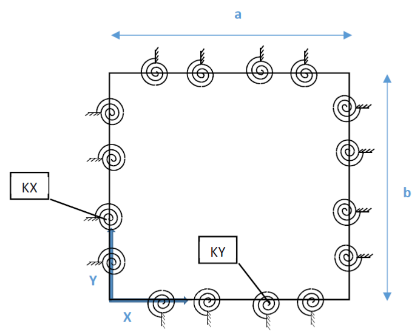

Figure 1. Geometry of rotationally restrained panel

2.4 *The Ritz method*

When using the Ritz method the solution is sought in the form of a series of arbitrary displacement functions that satisfy the essential boundary conditions. The geometric boundary conditions considered in the current work are

$$\begin{array}{ll} \text{At x=0, a} & \text{w=0} \\ \text{At y=0, b} & \text{w=0} \end{array} \tag{15}$$

Thus as the geometric boundary conditions are similar to that of a simply supported plate, the trigonometric functions used for simply supported boundary condition are used for restrained panels too. The functions used are.

$$\begin{aligned} w &= \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\ \varphi_x &= \sum_{m=1}^{M}\sum_{n=1}^{N} B_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\ \varphi_y &= \sum_{m=1}^{M}\sum_{n=1}^{N} C_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \end{aligned} \tag{16}$$

Where M and N are respectively the number of terms used in X and Y directions and A_{mn}, B_{mn} and C_{mn} are coefficients to be determined through the Ritz method.

Substituting equations (12), (13) and (14) in equation (11) we will have the general expression of the total potential energy in terms of displacements and by substituting equation (16) in that expression we will have the same expression in terms of A_{mn}, B_{mn} and C_{mn}. According to the Ritz method and the principle of minimum total potential energy the solution can be sought by minimizing the total potential energy with respect to the undetermined coefficients.

$$\frac{\partial \Pi}{\partial A_{m,n}} = 0\ , \ \frac{\partial \Pi}{\partial B_{m,n}} = 0, \frac{\partial \Pi}{\partial C_{m,n}} = 0 \tag{17}$$

which is a 3M×3N system of equations with a trivial solution. Setting the determinant of the coefficients of the system of equations to zero the critical buckling load and the mode shape associated to it will be determined. To do so for uniformly applied loads *{N₀}* is expressed as

$$\{N^0\} = \lambda\{n^0\} \tag{18}$$

where $\{n_0\} = [nx\ ny\ nxy]^T$ is the vector which defines the proportion of each load type in the pre-buckled state. In order to transform the system of linear equations (17) to the form a classical eigenvalue problem, by substituting equation (18) in equation (17) and rearranging the equations they can be rewritten in the form

$$\begin{gathered}\frac{\partial(U+U_r)}{\partial A_{mn}} + \lambda\frac{\partial V}{\partial A_{mn}} = 0\\ \frac{\partial(U+U_r)}{\partial B_{mn}} + \lambda\frac{\partial V}{\partial B} = 0\\ \frac{\partial(U+U_r)}{\partial C_{mn}} + \lambda\frac{\partial V}{\partial C_{mn}} = 0\end{gathered} \tag{19}$$

Putting these equations in a matrix form we will have

$$[K + \lambda K_s]\begin{Bmatrix}\{A_{mn}\}\\ \{B_{mn}\}\\ \{C_{mn}\}\end{Bmatrix} = \{0\}_{3M\times 3N} \tag{20}$$

where *K* is composed of the first terms in equation (18) and *Ks* is the matrix composed of the second terms of the same system of equations.

3 Results and discussion

In the current work in order to bring more generality to the investigations, the results are presented in the form of non-dimensional variables that define the buckling behavior of laminated curved plates [21]. The non-dimensional parameters that influence the buckling strength of a panel in terms of physical variables of buckling analysis are

$$\alpha = \frac{a}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4} \tag{21}$$

$$\beta = \frac{D_{12} + 2D_{66}}{\sqrt{D_{22}\,D_{11}}} \tag{22}$$

$$\gamma = \frac{D_{16}}{\left(D_{11}{}^3 D_{22}\right)^{1/4}} \tag{23}$$

$$\delta = \frac{D_{26}}{\left(D_{11} D_{22}{}^3\right)^{1/4}} \tag{24}$$

$$\bar{N}_x = \frac{N_x^0 b^2}{\pi^2 (D_{11} D_{22})^{1/2}} \tag{25}$$

$$\bar{N}_y = \frac{N_y^0 a^2}{\pi^2 (D_{11} D_{22})^{1/2}} \tag{26}$$

$$\bar{N}_{xy} = \frac{N_{xy}^0 b^2}{\pi^2 \left(D_{11} {D_{22}}^3\right)^{1/4}} \tag{27}$$

$$\bar{K}_x = \frac{aK_x}{D_{11}} \tag{28}$$

$$\bar{K}_y = \frac{aK_y}{D_{22}} \tag{29}$$

In the above, α is the non-dimensional aspect ratio, β is bending orthotropy and γ and δ are bending anisotropy parameters. $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ are the non-dimensional buckling coefficients. $\bar{K}x$ and $\bar{K}y$ are non-dimensional rotational stiffness applied at boundaries parallel to Y and X respectively.

For parametric studies, buckling of laminates with two stacking sequences (±45/±45/45)s and (±70/±70/70)s were investigated thoroughly. β, γ and δ of the (±70/±70/70)s laminate are 1.37, 0.07 and 0.166 respectively these values for the (±45/±45/45)s are 2.4, 0.22 and 0.22.

In the first step the effect of plate aspect ratio on the buckling strength of restrained plates was investigated. To do so the buckling loads were calculated for various values of non-dimensional variable α while keeping other non-dimensional values constant. To apply rotational restraint at the boundaries, rotational stiffnesses were applied in a way that $\bar{K}x$ and $\bar{K}y$ are equal at all edges. Figure 2 depicts the variation of the axial buckling coefficient versus α. Figure 3 shows the same results for buckling under shear stress. The behavior of the laminates under combined shear-axial loading were also investigated. This latter study was done for the specific case of α=1. Figure 4 depicts these results.

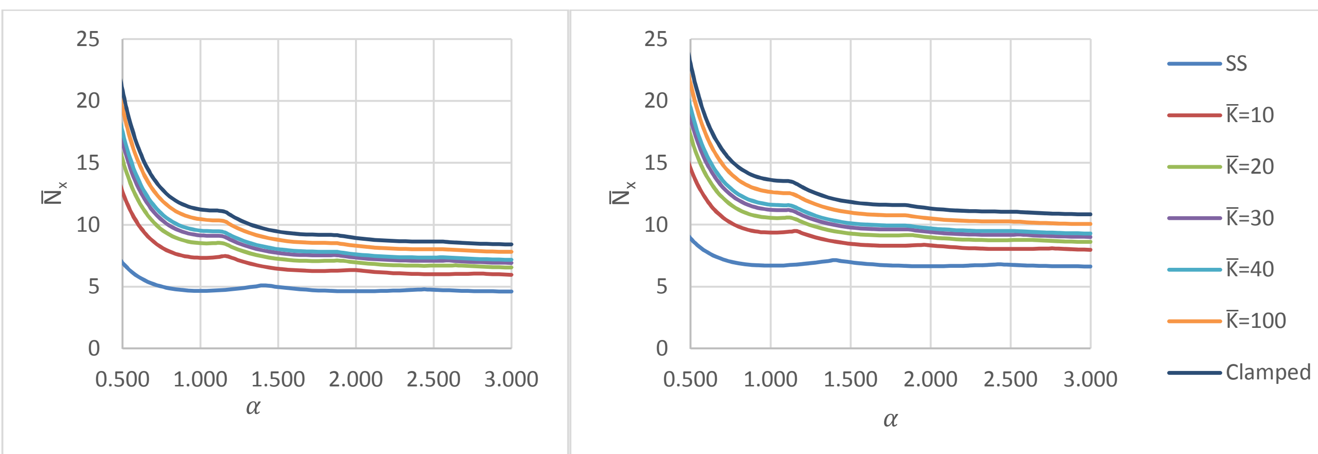

Figure 2. Variation of non-dimensional axial buckling load with non-dimensional aspect ratio for (±70/±70/70)s (left) and (±45/±45/45)s (right) laminates

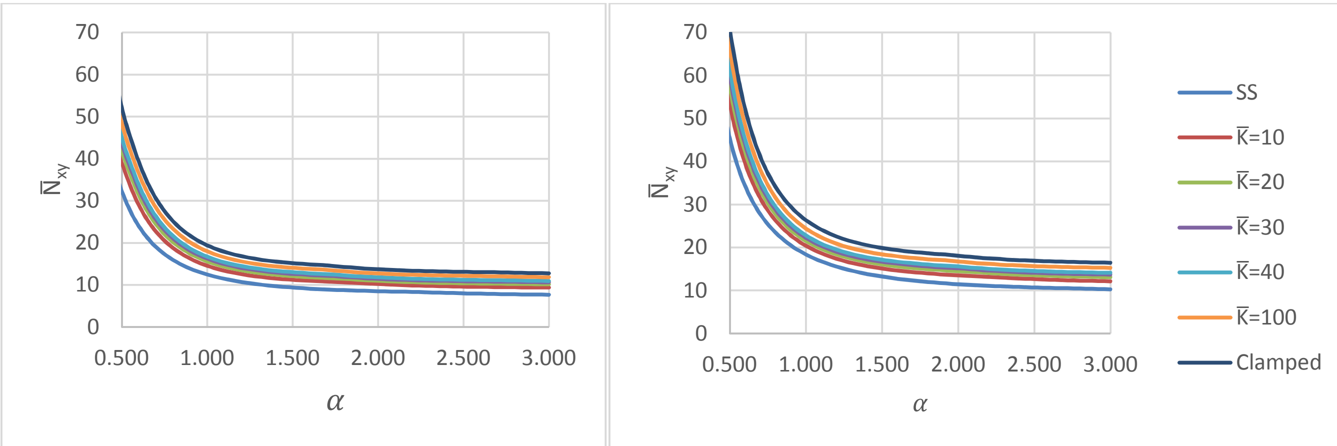

Figure 3. Variation of non-dimensional shear buckling load with non-dimensional aspect ratio for (±70/±70/70)s (left) and (±45/±45/45)s (right) laminates

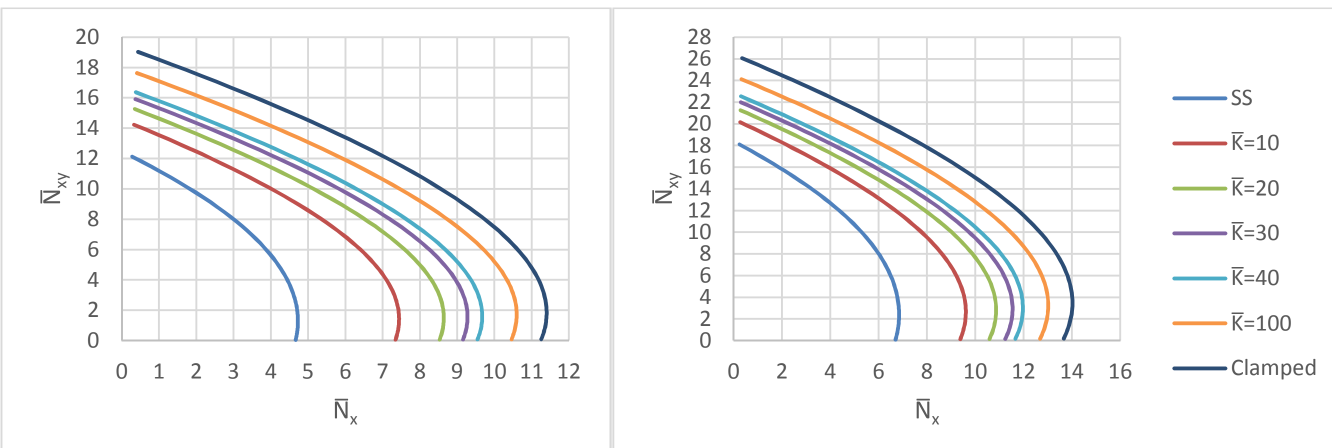

Figure 4. Interaction curve of shear and axial buckling for (±70/±70/70)s (left) and (±45/±45/45)s (right) laminates

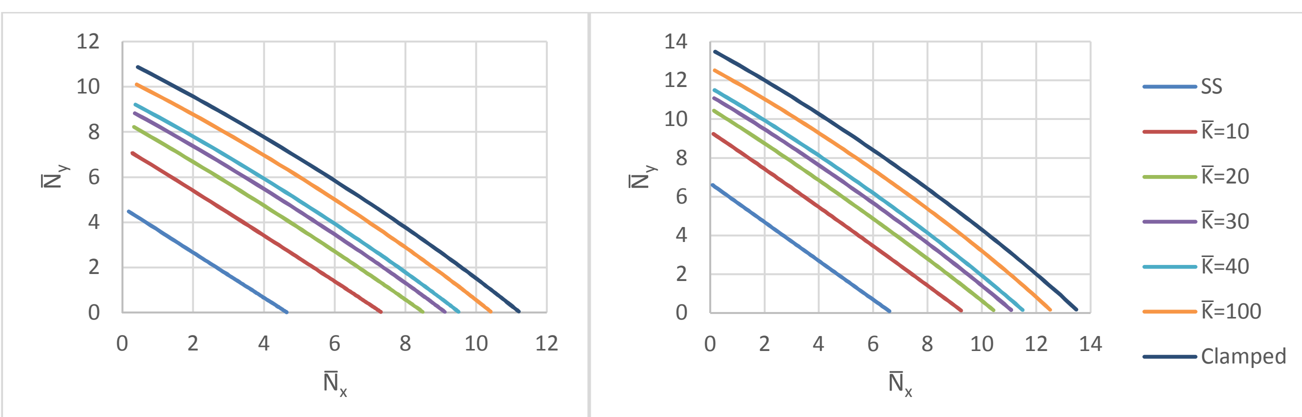

Figure 5. Interaction curve biaxial loading. Left (±70/±70/70)s and right (±45/±45/45)s laminates

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5 Concluding remarks

The present work has been an attempt to implement the Ritz method for modelling the buckling behavior of symmetrically laminated panels and also to investigate the effect of various parameters on the buckling load carrying capacity of rotationally restrained cylindrically curved panels. Through the derivation and parametric studies following points were concluded.

- In all cases studied it was observed that a non-dimensional boundary stiffness of about 600-700 is enough for simulating a fully clamped boundary condition and further increasing the stiffness may result in less accurate results due to numerical errors.
- By slightly increasing the boundary rotational stiffness, at first a sharp rise in the buckling load is observed, however for higher stiffness values this variation is less pronounced.
- The application of restraints not only changes the buckling critical load but also shifts the location of kink points on aspect ratio-buckling load graphs. In other words it affects the mode shapes experienced and the transition between different mode shapes.
- In the cases that were studied in the current research, it was noticed that the influence of rotational restraints on shear buckling is much less than their influence on axial buckling.
- The range of results obtainable using different stiffness values show that using simply supported boundary conditions in structural analyses may lead to drastic design conservatism.

6 REFERENCES

[1] G. J. Simitses and J. Giri. “Buckling of Rotationally Restrained Orthotropic Plates Under Uniaxial Compression,” vol. 11, pp. 345–364, 1977.

[2] E. Jaberzadeh and M. Azhari. “Elastic and inelastic local buckling of stiffened plates subjected to non-uniform compression using the Galerkin method,” *Appl. Math. Model.*, vol. 33, no. 4, pp. 1874–1885, 2009.

[3] R. Vescovini and C. Bisagni. “Single-mode solution for post-buckling analysis of composite panels with elastic restraints loaded in compression,” *Compos. Part B Eng.*, vol. 43, no. 3, pp. 1258–1274, 2012.

[4] Q. Chen and P. Qiao. “Shear buckling of rotationally-restrained composite laminated plates,” *Thin-Walled Struct.*, vol. 94, pp. 147–154, 2015.

[5] P. Qiao and G. Zou. “Local buckling of elastically restrained fiber-reinforced plastic plates and its application to box sections,” *J. Eng. Mech.*, vol. 128, no. 12, pp. 1324–1330, 2002.

[6] P. Qiao and G. Zou. “Local Buckling of Composite Fiber-Reinforced Plastic Wide-Flange

Sections," *J. Struct. Eng.*, vol. 129, no. 1, pp. 125–129, 2003.
[7] P. Qiao and L. Shan. "Explicit local buckling analysis and design of fiber-reinforced plastic composite structural shapes," *Compos. Struct.*, vol. 70, no. 4, pp. 468–483, 2005.
[8] C. Mittelstedt. "Stability behaviour of arbitrarily laminated composite plates with free and elastically restrained unloaded edges," *Int. J. Mech. Sci.*, vol. 49, no. 7, pp. 819–833, 2007.
[9] L. Shan and P. Qiao. "Explicit local buckling analysis of rotationally restrained composite plates under uniaxial compression." *Eng. Struct.*, vol. 30, no. 1, pp. 126–140, 2008.
[10] P. Qiao and X. Huo. "Explicit local buckling analysis of rotationally-restrained orthotropic plates under uniform shear," *Compos. Struct.*, vol. 93, no. 11, pp. 2785–2794, 2011.
[11] E. Villarreal and D. Abajo. "Buckling and modal analysis of rotationally restrained orthotropic plates," *Prog. Aerosp. Sci.*, vol. 78, pp. 116–130, 2015.
[12] D. G. Stamatelos, G. N. Labeas, and K. I. Tserpes. "Analytical calculation of local buckling and post-buckling behavior of isotropic and orthotropic stiffened panels," *Thin-Walled Struct.*, vol. 49, no. 3, pp. 422–430, 2011.
[13] L. E. Monterrubio. "Frequency and buckling parameters of box-type structures using the Rayleigh-Ritz method and penalty parameters," *Comput. Struct.*, vol. 104–105, pp. 44–49, 2012.
[14] J. L. Mantari and F. G. Canales. "Free vibration and buckling of laminated beams via hybrid Ritz solution for various penalized boundary conditions," *Compos. Struct.*, vol. 152, pp. 306–315, 2016.
[15] G. B. Chai, W. M. Banks, and J. Rhodes. "The instability behaviour of laminated panels with elastically rotationally restrained edges." *Compos. Struct.*, vol. 19, pp. 41–66, 1991.
[16] L. C. Bank and J. Yin. "Buckling of orthotropic plates with free and rotationally restrained unloaded edges," *Thin-Walled Struct.*, vol. 24, no. 1, pp. 83–96, 1996.
[17] J. W. Pizhong Qiao, Julio F. Davalos. "Local buckling of composite FRP shapes by discrete plate analysis," vol. 127, no. 3, pp. 245–255, 2001.
[18] T. Mizusawa and T. Kajita. "Vibration and buckling of rectangular plates with nonuniform elastic constraints in rotation," *Int. J. Solids Struct.*, vol. 23, no. 1, pp. 45–55, 1987.
[19] Q. Chen and P. Qiao. "Buckling analysis of laminated plate structures with elastic edges using a novel semi-analytical finite strip method," *Compos. Struct.*, vol. 152, pp. 85–95, 2016.
[20] J. M. Housner and M. Stein. "Numerical analysis and parametric study of the buckling of composite orthotropic compression and shear panels," 1975.
[21] M. P. Nemeth. "Nondimensional parameters and equations for nonlinear and bifurcation analyses of thin anisotropic quasi-shallow shells," *J. Appl. Mech.*, vol. 61, pp. 664–669, 1994.